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ON SOME NEW TENSORS AND THEIR PROPERTIES IN A FINSLER SPACE-I

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ABSTRACT

*In **(1990),** Certain new tensors in a Finsler space were introduced and studied by the author [4]. The purpose of the present paper is to define and study some new and very useful symmetric and skew-symmetric tensors of second and third order in a Finsler space of three dimensions. In the present paper, while studying three dimensional Finsler spaces the author has defined a new symmetric tensor D_{ijk}, which satisfies $D_{ijk}\, l^i = 0$ and $D_{ijk}g^{jk} = D_i = D\, n_i$. Besides defining this tensor and following the definition of C-reducible Finsler spaces Matsumoto [2] the author has further defined and studied D-reducible Finsler spaces. A symmetric tensor Q_{ijk} based on D_{ijk} and similar to P_{ijk} is also introduced and its relationship with other known tensors is studied. Several tensors corresponding to curvature tensor S_{ijkh} are also studied in F^3.*

KEYWORDS: *Three-Dimensional Finsler Space, D-Tensor, Q-Tensor*

INTRODUCTION

Let F^3be a three- dimensional Finsler space with the Moor's frame (l_i, m_i, n_i). Corresponding to this frame, the metric tensor and (h) hv-torsion tensors are given by Matsumoto [3] and Rund [6]

$$g_{ij} = l_il_j + m_im_j + n_in_j \tag{1.1}$$

And

$$C_{ijk} = C_{(1)}\, m_im_jm_k + C_{(2)}n_in_jn_k + \Sigma_{(ijk)}\, \{C_{(3)}\, m_in_jn_k - C_{(2)}\, m_im_jn_k\} \tag{1.2}$$

In equation (1.2), $\Sigma_{(ijk)}$ {}, represents cyclic permutation of indices i, j, k and summation. In a three- dimensional Finsler space h and v-covariant derivatives of a tensor are respectively given in Matsumoto [3]

$$K^h{}_{m/r} = \partial_rK^h{}_m - N^j{}_r\Delta_jK^h{}_m + K^k{}_mF_k{}^h{}_r - K^h{}_kF_m{}^k{}_r \tag{1.3}$$

And

$$K^h{}_{m//r} = \Delta_rK^h{}_m + K^j{}_mC_j{}^h{}_r - K^h{}_jC_m{}^j{}_r \tag{1.4}$$

where $\partial_r = \partial/\partial x^r$ and $\Delta_r = \partial/\partial y^r$.

From equations (1.3) and (1.4), we can easily obtain [3] 2

$$l^i{}_{/j} = 0,\ m^i{}_{/j} = n^ih_j, n^i{}_{/j} = -\ m^i\, h_j, \tag{1.5}$$

and

$L\, l^i_{//j} = h^i_j,\ L\, m^i_{//j} = -l^i m_j + n^i v_j,\ L\, n^i_{//j} = -l^i n_j - m^i v_j$ (1.6)

where $v_j = v_{2)3\gamma}\, e_{\gamma)j}$ and $h_j = H_{2)3\,\gamma} e_{\gamma)j}$.

The second and third curvature tensors in the sense of E. Cartan [1] are given by

$P_{ijkh} = \varsigma_{(i,j)}\ \{A_{jkh/i} + A_{ikr} P^r_{jh}\}$ (1.7)

$S_{ijkh} = \varsigma_{(h,k)}\{A_{ihr} A^r_{jk}\}$ (1.8)

Such that

$\varsigma_{(h,k)}\{P_{ijkh}\} = -\ S_{ijkh/\,0}$ (1.9)

where $\varsigma_{(h,k)}\{\}$ means interchange of h and k and subtraction.

SOME NEW TENSORS OF SECOND ORDER

Definition

In a Finsler space of three dimensions F^3, we define non-zero second order symmetric tensors $A_{ij}(x,y)$ and $B_{ij}(x,y)$ given by

$A_{ij}(x,y) = \sum_{(ij)}\{l_i m_j\},\ B_{ij}(x,y) = \sum_{(ij)}\{\ l_i n_j\}$ (2.1)

which satisfy

$A_{ij/\,k} = h_k B_{ij},\ B_{ij/\,k} = -\ h_k A_{ij},$ (2.2)

$A_{ij//k} = L^{-1}\{h_{ik} m_j + h_{jk}\, m_i - 2\, l_i\, l_j m_k + v_k B_{ij}\}$ (2.3)

and

$B_{ij//k} = L^{-1}\{h_{ik} n_j + h_{jk} n_i - 2 l_i l_j n_k - v_k A_{ij}\}$ (2.4)

From these equations we can obtain

Theorem

In a three- dimensional Finsler space F^3, tensors A_{ij} and B_{ij} satisfy

$A_{ij//k}\, l^i + L^{-1}(2\, l_j m_k - n_j v_k) = 0,\ B_{ij//k}\, l^i + L^{-1}(2 l_j n_k - m_j v_k) = 0,$ (2.5)

$\sum_{(ijk)}\{\ A_{ij/\,k} - h_k B_{ij}\} = 0,\ \sum_{(ijk)}\{\ B_{ij/\,k} + h_k A_{ij}\} = 0$ (2.6)

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$\sum_{(ijk)}\{\ A_{ij//k} - L^{-1}(2 h_{ij} m_k - 2 l_i l_j m_k + v_k B_{ij})\} = 0$ (2.7)

and

$\sum_{(ijk)}\{B_{ij//k} - L^{-1}(2\ h_{ij} n_k + 2 l_i l_j n_k - v_k A_{ij})\} = 0$ (2.8)

Definition

In a Finsler space of three dimensions F^3, we define non-zero second order symmetric tensors $T_{ij}(x,y)$ and $U_{ij}(x,y)$ given by

$T_{ij} = \sum_{(ij)}\{m_i n_j\}$, $U_{ij} = m_i m_j - n_i n_j$ (2.9)

These tensors satisfy

$T_{ij/k} = -2\, h_k U_{ij}$, $U_{ij/k} = 2\, h_k T_{ij}$, (2.10)

$T_{ij//k} = -L^{-1}\{l_i T_{jk} + l_j T_{ki} + 2\, v_k U_{ij}\}$ (2.11)

and

$U_{ij//k} = -L^{-1}(l_i U_{jk} + l_j U_{ki} - 2\, v_k T_{ij})$ (2.12)

which lead to

Theorem

In a three- dimensional Finsler space F^3, tensors T_{ij} and U_{ij} satisfy

$T_{ij//k}\, l^i = -L^{-1} T_{jk}$, $U_{ij//k}\, l^i = -L^{-1} U_{jk}$, $T_{ij//0} = 0$, $U_{ij//0} = 0$, (2.13)

$\sum_{(ijk)}\{ T_{ij/k} + 2\, U_{ij} h_k \} = 0$, $\sum_{(ijk)}\{ U_{ij/k} - 2\, h_k T_{ij} \} = 0$, (2.14)

$\sum_{(ijk)}\{ T_{ij//k} + 2\, L^{-1}(l_i T_{jk} + v_i U_{jk})\} = 0$, (2.15)

$\sum_{(ijk)}\{ U_{ij//k} + 2L^{-1}(l_i U_{jk} - v_i T_{jk})\} = 0$. (2.16)

Definition

In a Finsler space of three dimensions F^3, we define non-zero second order skew-symmetric tensors $E_{ij}(x,y)$, $F_{ij}(x,y)$ and $V_{ij}(x,y)$, given by

$E_{ij} = \varsigma_{(ij)}\{ l_i m_j\}$, $F_{ij} = \varsigma_{(ij)}\{l_i n_j\}$, $V_{ij} = \varsigma_{(ij)}\{m_i n_j\}$ (2.17)

These tensors satisfy

$E_{ij/k} = h_k F_{ij}$, $F_{ij/k} = -\, h_k E_{ij}$, $V_{ij/k} = 0$ (2.18)

$E_{ij//k} = L^{-1}(v_k F_{ij} - n_k V_{ij})$, $F_{ij//k} = L^{-1}(m_k V_{ij} - v_k E_{ij})$ (2.19)

$V_{ij//k} = L^{-1}(l_i V_{jk} + l_j V_{ki})$ (2.20)
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From equation (2.18), (2.19) and (2.20) we can obtain

Theorem

In a three -dimensional Finsler space F^3, Tensors E_{ij}, F_{ij} and V_{ij} satisfy

$E_{ij//k}\, l^i = L^{-1} v_k n_j$, $F_{ij//k}\, l^i = -\, L^{-1} v_k m_j$, $V_{ij//k}\, l^i = L^{-1} V_{jk}$ and

$\sum_{(ijk)}\{E_{ij/k} - h_k F_{ij}\} = 0$, $\sum_{(ijk)}\{ F_{ij\,k} + h_k E_{ij}\} = 0$, (2.21)

$\sum_{(ijk)}\{ E_{ij//k} - L^{-1} v_k F_{ij}\} = 0$, $\sum_{(ijk)}\{ F_{ij//k} + L^{-1} v_k E_{ij}\} = 0$ (2.22)

$\sum_{(ijk)}\{ V_{ij//k} - 2L^{-1} l_i V_{jk}\} = 0$. (2.23)

From equations (2.1), (2.8) and (2.17), we can easily establish

Theorem

In a three dimensional Finsler space F^3, tensors A_{ij}, B_{ij},T_{ij}, U_{ij}, $E_{ij}F_{ij}$ and V_{ij} satisfy $\varsigma_{(jk)}\{A_{hj}A_{ik}\} = E_{kj}E_{hi}$, $\varsigma_{(jk)}\{B_{hj} B_{ik}\}= F_{kj}F_{hi}$. $\varsigma_{(jk)}\{T_{hj}T_{ik}\}= V_{hi}V_{kj}= \varsigma_{(jk)}\{U_{hj}U_{ik}\}= \varsigma_{(jk)}\{h_{hj}h_{ik}\}$.

SOME NEW TENSORS OF THIRD ORDER

Definition

In a Finsler space of three- dimensions F^3, we define $\Delta_k A_{ij} = A_{ij.k}$, $\Delta_k B_{ij} = B_{ij.k}$, $\Delta_k T_{ij} = T_{ij.k}$ and $\Delta_k U_{ij} = U_{ij.k}$such that

$$A_{ij.k} = A_{ij//k} + m_r \sum_{(ij)}\{ l_j C^r_{ik}\} \quad (3.1)$$

$$B_{ij.k} =B_{ij//k} + n_r \sum_{(ij)}\{ l_j C^r_{ik}\} \quad (3.2)$$

$$T_{ij.k} =T_{ij//k} + m_r \sum_{(ij)}\{ n_j C^r_{ik}\} + n_r \sum_{(ij)}\{ m_j C^r_{ik}\} \quad (3.3)$$

$$U_{ij.k} = U_{ij//k} + m_r \sum_{(ij)}\{ m_j C^r_{ik}\} - n_r \sum_{(ij)}\{ n_j C^r_{ik}\} \quad (3.4)$$

These equations can be further solved with the help of

$$m_r C^r_{ik} = C_{(1)}\, m_i m_k - C_{(2)} T_{ik} + C_{(3)} n_i n_k \quad (3.5)$$

$$n_r C^r_{ik} = - C_{(2)}\, m_i m_k + C_{(3)} T_{ik} + C_{(2)} n_i n_k \quad (3.6)$$

From above equations we can obtain

Theorem

In a Finsler space of three - dimensions F^3, $A_{ij.0} = 0$, $B_{ij.0} = 0$, $T_{ij.0} = 0$, $U_{ij.0} = 0$.

Definition

In a Finsler space of three - dimensions F^3, we define $\Delta_k E_{ij} = E_{ij.k}$,

$\Delta_k F_{ij} = F_{ij.k}$ and $\Delta_k V_{ij} = V_{ij.k}$such that 5

$$E_{ij.k} = E_{ij//k} - m_r\, \varsigma_{(ij)}\, \{l_j C^r_{ik}\} \quad (3.7)$$

$$F_{ij.k} = F_{ij//k} - n_r\, \varsigma_{(ij)}\{ l_j C^r_{ik}\} \quad (3.8)$$

$$V_{ij.k} = V_{ij//k} + m_r\, \varsigma_{(ij)}\{n_j C^r_{ik} \}- n_r\, \varsigma_{(ij)}\, \{m_j C^r_{ik}\} \quad (3.9)$$

From above equations we can obtain

Theorem

In a Finsler space of three- dimensions F^3, $E_{ij.0} = 0$, $F_{ij.0} = 0$ and $V_{ij.0} = 0$.

THIRD ORDER SYMMETRIC TENSOR

D_{ijk}. Let D_{ijk} be a symmetric tensor in a Finsler space F^3 satisfying $D_{ijk}\, l^i = 0$, $D_{ijk} g^{jk} = D_i = D\, n_i$. Any tensor of the above type in F^3can be expressed as

$D_{ijk} = D_{(1)}\, m_i m_j m_k + D_{(2)} n_i n_j n_k + \Sigma_{(ijk)}\{D_{(3)}\, m_i m_j n_k + D_{(4)}\, m_i n_j n_k\}$ (4.1)

where $D_{(1)}$, $D_{(2)}$, $D_{(3)}$ and $D_{(4)}$ are scalars to be determined.

Multiplying equation (4.1) by g^{jk}, we obtain on simplification

$D\, n_i = (D_{(1)} + D_{(4)})\, m_i + (D_{(2)} + D_{(3)})\, n_i$,

which easily leads to

$D_{(2)} + D_{(3)} = D,\ D_{(1)} + D_{(4)} = 0.$ (4.2)

Thus equation (4.1) can also be expressed as

$D_{ijk} = D_{(1)}\, m_i m_j m_k + D_{(2)} n_i n_j n_k + \Sigma_{(ijk)}\{D_{(3)}\, m_i m_j n_k - D_{(1)}\, m_i n_j n_k\}$, (4.3)

which leads to the following :

Definition

In a three -dimensional Finsler space F^3, the symmetric tensor D_{ijk} satisfying $D_{ijk}\, l^i = 0$, $D_{ijk} g^{jk} = D\, n_i$ and given by equation (4.3) shall be called a D-tensor.

Remarks

It is to be noticed that D_{ijk}, which looks similar to C_{ijk}exists for $n \geq 3$ only.

Equation (4.3) can alternatively be expressed as

$D_{ijk} = \Sigma_{(ijk)}\{X_k\, m_i m_j + Y_k n_i n_j\}$ (4.4)

where

$X_k = (1/3)\, D_{(1)} m_k + D_{(3)} n_k$ (4.5)

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and

$Y_k = (1/3)\, D_{(2)} n_k - D_{(1)} m_k$ (4.6)

Equation (4.3) can also be expressed as

$D_{ijk} = \Sigma_{(ijk}\{X_{ij} m_k + Y_{ij} n_k\}$ (4.7)

where $X_{ij} = X_i m_j$ and $Y_{ij} = Y_i n_j$.

Now we shall consider some special cases.

Case I

If we assume $D_{ijk} = 0$, equation (4.3) on simplification gives $D_{(2)} = - D_{(3)}$. Conversely, if we assume $D_{(2)} = - D_{(3)}$, in equation (4.3), it gives $D_i = 0$. Hence we have:

Theorem

The necessary and sufficient condition for the vector D_i to vanish in F^3 is given by $D_{(2)} = - D_{(3)}$.

Case II. If in a special case we assume that $D_{(1)} = 0$, $D_{(3)} = 0$, $D_{(2)} = D$, equation (4.3) can be expressed as

$$D_{ijk} = D^{-2} D_i D_j D_k \tag{4.8}$$

Conversely, if we assume (4.8), it leads to $D_{(1)} = 0$, $D_{(3)} = 0$, $D_{(2)} = D$. Hence we have:

Theorem

The necessary and sufficient condition for D_{ijk} to be expressed as in (4.7), in a three dimensional Finsler space F^3, is that $D_{(1)} = 0$, $D_{(3)} = 0$, $D_{(2)} = D$.

Case III

If $D_{(1)} = 0$, $D_{(2)} = D_{(3)} = D/2$, equation (4.3) gives

$$D_{ijk} = (D/2)[n_i n_j n_k + \Sigma_{(ijk)}\{m_i m_j n_k\}] \tag{4.9}$$

Hence we have:

Theorem

In a three dimensional Finsler space F^3, if $D_{(1)} = 0$, $D_{(2)} = D_{(3)} = D/2$, D_{ijk} is given by (4.9).

Case IV. If $D_{(1)} = 0$, $D_{(2)} = 0$, $D_{(3)} = D$, equation (4.3) gives

$$D_{ijk} = D\, \Sigma_{(ijk)}\{m_i m_j n_k\} \tag{4.10}$$

Conversely, if D_{ijk} is given by (4.10), equation (4.3) gives $D = D_{(3)}$, $D_{(2)} = 0$ and

$$D_{(1)}\, [m_i m_j m_k - \Sigma_{(ijk)}\{m_i n_j n_k\}] = 0 \tag{4.11}$$

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If we multiply equation (4.11) by m^i, we get $D_{(1)}U_{jk} = 0$ and if we multiply equation (4.11) by n^i, we get $D_{(1)}T_{jk} = 0$. As we know that neither U_{jk} nor T_{jk} vanish, therefore $D_{(1)} = 0$. Hence we have:

Theorem

In a three dimensional Finsler space F^3, the necessary and sufficient condition for $D_{(1)} = 0$, $D_{(2)} = 0$ and $D_{(3)} = D$, is that D_{ijk} is represented by (4.10)

PROPERTIES OF D-TENSOR IN F^3

Let $D_{ijk}m^k = {'D_{ij}}$ and $D_{ijk}n^k = {*D_{ij}}$, then from equation (4.3) similar to Shimada [7] and Rastogi [5], we can obtain

$$'D_{ij} = D_{(1)} U_{ij} + D_{(3)} T_{ij} \tag{5.1}$$

and

$$*D_{ij} = D_{(2)} n_i n_j – D_{(1)} T_{ij} + D_{(3)}\, m_i m_j \tag{5.2}$$

Remark

If we take $'D_{ij} = 0$, equation (5.1) gives $D_{(1)} = 0$, $D_{(3)} = 0$ and $D_{(2)} = D$, while for $*D_{ij} = 0$, $D_{(1)} = 0$, $D_{(3)} = 0$ and $D_{(2)} = 0$.

From equations (5.1) and (5.2) we can observe that tensors $'D_{ij}$ and $*D_{ij}$ are symmetric in i and j and also satisfy $'D_{ij}g^{ij} = 0$ and $*D_{ij}g^{ij} = D$. Hence we have:

Theorem

In a three- dimensional Finsler space F^3, the tensors $'D_{ij}$ and $*D_{ij}$ satisfy $'D_{ij}g^{ij} = 0$ and $*D_{ij}g^{ij} = D$.

Further from equations (5.1) and (5.2) we can obtain

$$'D_{ij}m^j = D_{(1)} m_i + D_{(3)}n_i,\ 'D_{ij}m^j m^i = D_{(1)},\ 'D_{ij}m^jn^i = D_{(3)} \quad (5.3)$$

$$*D_{ij}m^j = D_{(3)} m_i – D_{(1)}n_i,\ *D_{ij}m^j m^i = D_{(3)},\ *D_{ij}m^jn^i = -D_{(1)} \quad (5.4)$$

Using equation (1.3), from equation (5.1) and (5.2) we can obtain

$$'D_{ij/ k} = \{(D_{(3)/ k} + 2D_{(1)}h_k\} T_{ij} + \{D_{(1) / k}- 2 D_{(3)}h_k\}V_{ij} \quad (5.5)$$

and

$$*D_{ij/ k} = (D_{(2) / k} – 2 D_{(1)}h_k) n_in_j + (D_{(3) / k} + 2 D_{(1)}h_k) m_im_j$$

$$- (D_{(1) / k} + D_{(2)} h_k – D_{(3)}h_k) T_{ij} \quad (5.6)$$

From equations (5.5) and (5.6), we can obtain

$$'D_{ij/ k}l^k = \{(D_{(3)/ 0} + 2D_{(1)} h_0\} T_{ij} + \{D_{(1) / 0}- 2 D_{(3)} h_0\}U_{ij} \quad (5.7)$$

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$$*D_{ij/ k}l^k= (D_{(2) / 0} – 2 D_{(1)} h_0) n_in_j + (D_{(3) / 0} + 2 D_{(1)} h_0) m_im_j$$

$$- (D_{(1) / 0} + D_{(2)} h_0 – D_{(3)} h_0) T_{ij} \quad (5.8)$$

and

$$'D_{ij/ k}l^j = 0,\ *D_{ij/ k}l^j = 0. \quad (5.9)$$

Hence we have:

Theorem

In a three -dimensional Finsler space F^3, h-covariant derivatives of tensors $'D_{ij}$ and $*D_{ij}$ satisfyequations (5.7), (5.8) and (5.9).

Using equation (1.4), from equation (5.1), we can obtain

$$'D_{ij//k} = \{D_{(3)//k} + 2 v_k L^{-1} D_{(1)}\} T_{ij} + \{D_{(1) //k}- 2 v_k L^{-1}D_{(3)}\}U_{ij}.$$

$$- A_{ij} L^{-1}(D_{(3)}n_k + D_{(1)}m_k) - B_{ij} L^{-1}(D_{(3)}m_k – D_{(1)}n_k) \quad (5.10)$$

and

$*D_{ij//k} = \{D_{(3)//k} + 2L^{-1} D_{(1)}v_k\}m_i m_j + (D_{(2)//k} - 2L^{-1}D_{(1)}v_k) n_i n_j$

$- \{D_{(1)//k} + L^{-1}(D_{(2)}-D_{(3)}) v_k\} T_{ij} + L^{-1}[(D_{(1)}m_k - D_{(2)}n_k) B_{ij}$

$- (D_{(3)}m_k - D_{(1)}n_k) A_{ij}]$ (5.11)

From equations (5.10) and (5.11), we can obtain

$'D_{ij//k}l^j = - L^{-1}[D_{(3)}T_{ij} + D_{(1)} U_{ij}]$ (5.12)

$'D_{ij//0} = D_{(3)//0} T_{ij} + D_{(1)//0}U_{ij}$ (5.13)

and

$*D_{ij//k}l^j = L^{-1}[D_{(1)}T_{ij} - D_{(2)}n_i n_k - D_{(3)} m_i m_k]$ (5.14)

$*D_{ij//0} = D_{(2)//0}n_i n_j - D_{(1)//0}T_{ij} + D_{(3)//0} m_i m_j$ (5.15)

From equations (5.1) and (5.12), we can obtain

$'D_{ij//k}l^j + L^{-1'} D_{ik} = 0$ (5.16)

Hence we have:

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Theorem

In a three- dimensional Finsler space F^3, the tensor $^{'}D_{ik}$ satisfies equation (5.16).

Similarly from equations (5.2) and (5.14), we can obtain

$*D_{ij//k}l^j + L^{-1'*}D_{ik} = 0$ (5.17)

Hence we have:

Theorem

In a three - dimensional Finsler space F^3, the tensor $^{'*}D_{ik}$ satisfies equation (5.17).

D-REDUCIBLE FINSLER SPACES

Following calculations similar to the one used by Matsumoto [2], **i**n analogy to the definition of C-reducible Finsler space, we here define D-reducible Finsler space.

Definition

A Finsler space F^3, shall be called D-reducible Finsler space if the tensor D_{ijk} is defined as

$D_{ijk} = (1/4) \sum_{(ijk)}\{h_{ij} D_k\}$ (6.1)

Example. Let us consider a Finsler space F^3, in which $D_{(1)} = 0$, $D_{(2)} = (3/4) D$ and $D_{(3)} = (1/4) D$, then the tensor D_{ijk} is expressible as

$D_{ijk} = D [(3/4) n_i n_j n_k + (1/4) \Sigma_{(ijk)}\{m_i m_j n_k\}]$ (6.2)

Equation (6.2) is nothing but equation (6.1) written in alternative form.

From equation (6.2), we can obtain by virtue of equation (3.1)

$'D_{ij} = (1/4)\, D\, T_{ij}$ (6.3)

$*D_{ij} = (1/4)\, D\, (h_{ij} + 2\, n_i n_j)$ (6.4)

Using equations (1.3) and (4.3) we can obtain on simplification

$D_{ijk\,/\,r} = \{D_{(1)\,/\,r} - 3\, D_{(3)} h_r\}\, m_i m_j m_k + (D_{(2)\,/\,r} - 3D_{(1)} h_r)\, n_i n_j n_k$

$+\, \Sigma_{(ijk)}[\{(D_{(3)\,/\,r} + 3\, D_{(1)} h_r\}\, m_i m_j n_k - \{(D_{(1)/\,r} + D_{(2)} h_r - 2\, D_{(3)} h_r)\}\, m_i n_j n_k]$ (6.5)

If we define $Q_{ijk} = D_{ijk\,/\,0}$, from equation (6.5), we get

$Q_{ijk} = \{D_{(1)\,/\,0} - 3\, D_{(3)}\, h_0\}\, m_i m_j m_k + (D_{(2)\,/\,0} - 3D_{(1)} h_0)\, n_i n_j n_k$

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$+\, \Sigma_{(ijk)}[\{D_{(3)\,|\,0} + 3\, D_{(1)} h_0\}\, m_i m_j n_k - \{D_{(1)|\,0} + (D_{(2)} - 2\, D_{(3)}) h_0)\}\, m_i n_j n_k]$ (6.6)

If we assume that the tensor Q_{ijk} is proportional to D_{ijk}, i.e., $Q_{ijk} = \lambda\, D_{ijk}$, where λ is a scalar and is coefficient of proportionality, from equations (4.3) and (6.6), by comparing coefficients, we can obtain

- $D_{(1)\,/\,0} - 3\, D_{(3)}\, h_0 = \lambda\, D_{(1)}$, (ii) $D_{(2)\,/\,0} - 3D_{(1)} h_0 = \lambda\, D_{(2)}$,
- $D_{(3)\,/\,0} + 3\, D_{(1)} h_0 = \lambda\, D_{(3)}$, (iv) $D_{(1)/0} + D_{(2)} h_0 - 2\, D_{(3)}\, h_0 = \lambda\, D_{(1)}$

From (i) and (iv), we can obtain $(D_{(2)} + D_{(3)}) h_0 = 0$ and from (ii) and (iii), we get

$D_{(2)\,/\,0} + D_{(3)\,/\,0} = \lambda\, (D_{(2)} + D_{(3)})$. Since $h_0 \neq 0$, it implies $D_{(2)} + D_{(3)} = 0$, i.e., $D = 0$,

which will imply $D_i = 0$ and $D_{ijk} = 0$. Hence we have:

Theorem

In a three - dimensional Finsler space, if we assume that $Q_{ijk} = \lambda\, D_{ijk}$,

then both Q_{ijk} and D_{ijk} vanish.

From equation (6.6) by virtue of $Q_{ijk} g^{jk} = Q_i$, we can obtain $Q_i = D_{\,/\,0} n_i -$

$D\, h_0\, m_i$. For a D-reducible Finsler space F^3, by virtue of (6.2), the tensor Q_{ijk} can

be expressed as

$Q_{ijk} = -\, (1/4)\, D\, h_0 \sum_{(ijk)} \{h_{ij} m_k\}$ (6.7)

From the definition of C-reducible Finsler space F^3 and equation (6.7), we can obtain

$C\, Q_{ijk} + D\, h_0 C_{ijk} = 0$. Hence we have:

Theorem

In a C-reducible Finsler space F^3, torsion tensor C_{ijk}, and in a D-reducible Finsler space F^3 torsion tensor Q_{ijk}

satisfy $C\, Q_{ijk} + D\, h_0 C_{ijk} = 0$.

Similar to the definition of P-reducibility, we can give following:

Definition

A Finsler space F^3, shall be called Q-reducible if the Q-tensor is given by

$$Q_{ijk} = (1/4) \sum_{(ijk)}\{h_{ij} Q_k\} \quad (6.8)$$

Equation (6.8) can alternatively be expressed as

$$Q_{ijk} = (D_{/0}/4)\,(h_{ij}n_k + h_{jk}n_i + h_{ki}n_j) - (Dh_0/4)(h_{ij}m_k + h_{jk}\, m_i + h_{ki}m_j) \quad (6.9)$$

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If a Q-reducible space is both C-reducible and D-reducible, from equation (6.9), we can obtain $CDQ_{ijk} = -D^2\, h_0 C_{ijk} + D_{/0} C\, D_{ijk}$. Hence we have:

Theorem

A three - dimensional Q-reducible Finsler space F^3, which is both D-reducible and C-reducible satisfies $CDQ_{ijk} = -D^2\, h_0 C_{ijk} + D_{/0} C\, D_{ijk}$.

TENSOR D'_{ijkh}.

In analogy to the definition of v-curvature tensor S_{ijkh} based on torsion tensor C_{ijk}, we define here the tensor D'_{ijkh} based on D_{ijk} as follows:

$$D'_{ijkh} = D_{ihr} D^r_{\ jk} - D_{ikr} D^r_{\ jh} \quad (7.1)$$

Substituting the value of D_{ijk} from equation (2.2) in (7.1), we obtain on simplification

$$D'_{ijkh} = (2D_{(1)}{}^2 - D_{(2)}\, D_{(3)} + D_{(3)}{}^2)\,(m_k n_h - m_h n_k)\,(m_i n_j - m_j n_i) \quad (7.2)$$

We know that in F^3, the tensor $h_{ik}h_{jh} - h_{ih}h_{jk} = (m_i n_j - m_j n_i)\,(m_k n_h - m_h n_k)$, therefore equation (7.2) can also be expressed as

$$D'_{ijkh} = (2D_{(1)}{}^2 - D_{(2)}\, D_{(3)} + D_{(3)}{}^2)\,(h_{ik}h_{jh} - h_{ih}h_{jk}) \quad (7.3)$$

Following proposition (29.2) of Matsumoto [2], we can obtain

$$D'_{ijkh} = D^*\,(h_{ik}h_{jh} - h_{ih}h_{jk}) \quad (7.4)$$

where D^* is a (0)p-homogeneous scalar satisfying

$$D^* = (2\, D_{(1)}{}^2 - D_{(2)}\, D_{(3)} + D_{(3)}{}^2) \quad (7.5)$$

Hence we have:

Theorem

In a three- dimensional Finsler space F^3, there exists a tensor D'_{ijkh} given by (7.2) such that its scalar D^* is given by (7.5).

From equation (7.2), we can observe that $D'_{ijkh} = 0$ implies either $h_{ij} = 0$, i.e., F^3 is a Riemannian space or $(2 D_{(1)}^2 - D_{(2)} D_{(3)} + D_{(3)}^2) = 0$. Hence we have:

Theorem

In a three- dimensional non-Riemannian Finsler space F^3, the necessary and sufficient condition for the tensor D'_{ijkh} to vanish is given by $D^* = 0$.

From $D'_{ijkh} g^{jh} = D'_{ik}$ and $D'_{ik} g^{ik} = D'$, we can obtain from equation (7.3)

$D'_{ij} = D^* h_{ij}$, $D' = 2D^*$. Hence we have:

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Theorem

In a three- dimensional non-Riemannian Finsler space D'-Ricci tensorand

D'-scalar satisfy$D'_{ij} = D^* h_{ij}$, $D' = 2D^*$.

In a D-reducible Finsler space equation (7.2) is given by

$$D'_{ijkh} = -(1/8) D^2 (h_{ik} h_{jh} - h_{ih} h_{jk}) \tag{7.6}$$

which implies $D'_{ij} = -(1/8) D^2 h_{ij}$ and $D' = -(1/4) D^2$.

TENSOR Q_{ijkh}

Corresponding to tensor Q_{ijk}, we define tensor Q_{ijkh} as follows:

$$Q_{ijkh} = Q_{ihr} Q^r_{jk} - Q_{ikr} Q^r_{jh} \tag{8.1}$$

Substituting value of Q_{ihr} etc. from equation (4.3) in (8.1) and solving we get

$$Q_{ijkh} = (A_3^2 + A_4^2 + A_1 A_4 - A_2 A_3)(m_j n_i - m_i n_j)(m_h n_k - m_k n_h), \tag{8.2}$$

where

$$A_1 = D_{(1)/0} - 3D_{(3)} h_0,\ A_2 = D_{(2)/0} - 3 D_{(1)} h_0$$

$$A_3 = D_{(3)/0} + 3D_{(1)} h_0,\ A_4 = D_{(1)/0} + (D_{(2)} - 2 D_{(3)}) h_0 \tag{8.3}$$

From equation (8.3), we can observe that

$$A_4 - A_1 = (D_{(1)} + D_{(3)}) h_0,\ A_2 + A_3 = D_{(2)/0} + D_{(3)/0} \tag{8.4}$$

Hence we have:

Theorem

In a three- dimensional Finsler space F^3, tensor Q_{ijkh} is given by(8.2), such that its coefficients satisfy (8.4).

Comparing equations (7.2) and (8.2), we can obtain

$$Q_{ijkh} = (A_3^2 + A_4^2 + A_1 A_4 - A_2 A_3)(2D_{(1)}^2 - D_{(2)} D_{(3)} + D_{(3)}^2)^{-1} D_{ijkh} \tag{8.5}$$

Hence we can obtain

Theorem

In a three -dimensional Finsler space F^3, the tensors Q_{ijkh} and D_{ijkh} are proportional to each other.

For a D-reducible Finsler space, $A_1 = - (3/4) D h_0$, $A_2 = 0$, $A_3 = 0$, $A_4 = (1/4) Dh_0$, therefore equation (8.2) gives

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$$Q_{ijkh} = - (1/8) D^2 h_0^2 (m_j n_i - m_i n_j)(m_h n_k - m_k n_h) \quad (8.6)$$

which implies $Q_{ijkh} g^{jh} = Q_{ik} = - (1/8) D^2 h_0^2 h_{ik}$ and $Q_{ik} g^{ik} = - (1/4) D^2 h_0^2$.

Hence we have:

Theorem

In a three -dimensional D-reducible Finsler space F^3, tensor Q_{ijkh} is expressed by equation (8.6).

For a Q-reducible Finsler space F^3, $A_1 = - (3/4) D h_0$, $A_2 = (3/4) D_{/0}$, $A_3 = (1/4) D_{/0}$ and $A_4 = (1/4) D h_0$, therefore equation (8.2) can be expressed as

$$Q_{ijkh} = - (1/8)(D_{/0}^2 + D^2 h_0^2) (m_j n_i - m_i n_j)(m_h n_k - m_k n_h) \quad (8.7)$$

Hence we have:

Theorem

In a three- dimensional Q-reducible Finsler space F^3, tensor Q_{ijkh} is expressed by equation (8.7).

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